Allometric scaling: blood volume derivation

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As written in the article, the total volume of blood in an organism is given by the following sum:

\[ V_b = \sum_{k=0}^{N} V_k N_k \tag{1} \]

where \( V_b \) is the volume of a tube in the \( k^{th} \) level of the network and \( N_k \) is the number of tubes at that level. First, we can rewrite Eq. 1 as

\[ V_b = \sum_{k=0}^{N} \pi r_k^2 l_k m_k \tag{2} \]

where we have replaced \( V_k \) with the volume of a cylinder with radius \( r_k \) and length \( l_k \), and we have replaced \( N_k = m_k \) as the number of tubes at the \( k^{th} \) level, since \( N_k = \prod_{i=0}^{k} m_i = m^k \) since \( m_i = m \) for every network level.

Now, we use the ratio between successive radii and lengths to write the quantities \( r_k \) and \( l_k \) in terms of \( \beta \), \( \gamma \), \( r_{cap} \), and \( l_{cap} \), all of which are invariant across organisms. Recalling that \( \beta_k = r_{k+1}/r_k = \beta \) and \( \gamma_k = l_{k+1}/l_k = \gamma \), we can write

\[ r_k = \beta^{-1} r_{k+1} = \beta^{-2} r_{k+2} = \ldots = \beta^{-n} r_{k+n} \]
\[ l_k = \gamma^{-1} l_{k+1} = \gamma^{-2} l_{k+2} = \ldots = \gamma^{-n} l_{k+n} \]

if we set \( n + k = N \), i.e. we carry out this recursive relationship all the way up to level \( N \), the level of capillaries, we can write Eq. 2 as

\[ V_b = \sum_{k=0}^{N} \pi (\beta^{k-N} r_{cap})^2 (\gamma^{k-N} l_{cap}) m^k = \pi \beta^{-2N} \gamma^{-N} r_{cap}^2 l_{cap} \sum_{k=0}^{N} (\beta^2 \gamma m)^k \]

Rewriting \( a = \pi \beta^{-2N} \gamma^{-N} r_{cap}^2 l_{cap} \) and \( x = \beta^2 \gamma m \neq 1 \), we have a geometric series

\[ \sum_{k=0}^{N} a x^k = a \frac{1 - x^{N+1}}{1 - x} \]

\[ \rightarrow V_b = (\pi r_{cap}^2 l_{cap}) (\beta^2 \gamma)^{-N} \frac{1 - (\beta^2 \gamma m)^{N+1}}{1 - \beta^2 \gamma m} \]
\[ V_b = V_{cap} \left( \beta^2 \gamma \right)^{-N} \frac{(\beta^2 \gamma m)^{N+1}}{\beta^2 \gamma m} - (\beta^2 \gamma m)^{-(N+1)} - 1 \]
\[ V_b = V_{cap} N_{cap} \frac{(\beta^2 \gamma m)^{-(N+1)} - 1}{(\beta^2 \gamma m)^{-1} - 1} \]

where we have replaced \( V_{cap} = \pi r_{cap}^2 l_{cap} \) and \( N_{cap} = m^N \). If we then make an approximation using \( \beta^2 \gamma m < 1 \) and \( N \gg 1 \),
\[ V_b \approx V_{cap} N_{cap} \frac{(\beta^2 \gamma m)^{-N}}{1 - \beta^2 \gamma m} \]

\[ V_b \propto (\beta^2 \gamma)^{-N} \]

since \( N_{cap} \) cancels out and all other terms (\( V_{cap}, \beta, \gamma, m \)) are constant across organisms.